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Diego Napp Avelli and Harry L. Trentelman

***Abstract*— The notion of interconnection is the basis of control in the behavioral approach. In this setting, feedback interconnection of systems is based on the still more fundamental concept of regular interconnection, which has been introduced by J. C. Willems. In this paper, the following problem is addressed: given a plant, under what conditions does there exist a controller such that their interconnection is regular and has finite codimension with respect to a certain desired system. If so, provide a constructive solution to the problem. The second part of the paper treats the related problem of decomposition of systems. First, the autonomous/controllable decomposition is studied, and finally we look at the decomposition of the controllable part.**

keywords: multidimensional systems, behavioral approach regular interconnection, feedback interconnection, multivariable polynomial modules, decomposition , controllability, strong controllability

I. INTRODUCTION

The behavioral approach relies on the idea that systems are described by equations, but their properties are naturally described in terms of the set of all solutions to the equations. This is formalized by the notion of system behavior due to J.C.Willems, and denoted by $\mathfrak{B}$. In this setting, a new perspective to control is given, see [17], based on interconnection of systems, where no *a priori* input/output partition is considered. The act of controlling a system is simply viewed as intersecting its behavior $\mathfrak{B}$ with a controller behavior $\mathfrak{B}_c$ in order to achieve a desired behavior $\mathfrak{B}_d = \mathfrak{B} \cap \mathfrak{B}_c$. Of particular interest is the interconnection, called regular interconnection, where the restrictions imposed on the plant by the controller are not redundant, i.e. the restrictions of the controller are independent of the restrictions already present in the plant. Hence the notion of feedback control, which is of significant interest in modern control theory, is based on the still more fundamental concept of regular interconnection. It is, indeed, a simple example of regular interconnection since the controller imposes restrictions only on the plant input, which is not restricted by the plant.

The regular interconnection problem can be formulated as follows: given a plant behavior $\mathfrak{B}$ together with a desired behavior, find if possible, another behavior (the controller) such that the interconnection is regular and equal to the given desired behavior.

J.C.Willems in [17] stated and solved this problem for one dimensional behaviors. The multidimensional counterpart was treated by P. Rocha and J.Wood [11], [10], E. Zerz [20], and H. L. Trentelman and D. Napp [13].

Conditions and an algorithm were given for solving the problem. Actually, for multivariable behaviors, these conditions are very seldomly satisfied and strong properties will be required on the plant and the desired system.

Therefore, the limits of achievability by regular interconnection can be further studied. This suggests the idea of looking for an equivalent problem with weaker requirements. In this paper we will treat the following problem:

Given a plant behavior $\mathfrak{B}$ and a certain desired behabior $\mathfrak{B}_d$, find if possible, another behavior (the controller) such that the interconnection is regular and is contained in the given desired behavior with finite codimension, i.e. find if possible another behavior $\mathfrak{B}_c$ such that the interconnection is regular and $\mathfrak{B}_d/\mathfrak{B}\cap\mathfrak{B}_c$ is an autonomous behavior that is finite-dimension as a vector space over a field k, see [4], [8] (we also use the notation $dim_k(\mathfrak{B}_d/\mathfrak{B}\cap\mathfrak{B}_c) < \infty$ to denote that it is finite-dimension over the field k). In the 1D case, all autonomous behaviors are finite-dimensional, which means that the state space is finite dimensional. For multivariable behaviors this is, in general, not longer true, since it could have an infinite set of initial conditions. These special class of autonomous behaviors are called *strongly autonomous* in [9].

If such $\mathfrak{B}_c$ exists then we say that $\mathfrak{B}_d$ is almost achievable by regular interconnection from $\mathfrak{B}$. This constitutes a generalization of the regular interconnection problem as it represents the 'closest' achievability one can get through regular interconnection in the sense of finite dimension.

Furthermore, in this paper we investigate in some detail the related problem of decomposing a given behavior into the sum of finer components. It is immediately apparent that decomposition is a powerful tool for the analysis of the system properties. Decomposition is, indeed, of particular interest in the case of multidimensional systems, where a description of the nD systems trajectories can be complicated and decomposing the original behavior into smaller components seems to be an effective way for simplifying the systems analysis.

The autonomous-controllable decomposition has played a significant role in the theory of linear time-invariant systems. Such decomposition expresses the idea that every trajectory of the behavior can be thought of as the sum of two components: a free evolution, only depending on the set of initial conditions, and a forced evolution, due to the presence of the input. In the case of $1D$ systems, this sum is direct,

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i.e.

$$\mathfrak{B} = \mathfrak{B}_{cont} + \mathfrak{B}_{aut} \text{ and } \mathfrak{B}_{cont} \cap \mathfrak{B}_{aut} = 0.$$

Here $\mathfrak{B}_{cont}$ and $\mathfrak{B}_{aut}$ represent the controllable and autonomous part of $\mathfrak{B}$, respectively.
However, this decomposition is, in general, not longer direct for $n \geq 2$, and we may have that the controllable part of $\mathfrak{B}$, (which is uniquely defined for a given $\mathfrak{B}$) intersects all possible autonomous parts involved in the controllable-autonomous decomposition [19], [15], [3].

Finally, in our quest to completely decompose a behavior, we address the problem of decomposing the controllable part. The following problem is studied: Given a controllable behavior $\mathfrak{B}$ and a sub-behavior $\mathfrak{B}_a \subset \mathfrak{B}$, find a third behavior $\mathfrak{B}_b \subset \mathfrak{B}$ such that $\mathfrak{B}_b + \mathfrak{B}_a = \mathfrak{B}$ and $\mathfrak{B}_b \cap \mathfrak{B}_a$ has finite dimension.

If such $\mathfrak{B}_c$ exists, then we say that $\mathfrak{B}_a$ is an almost direct summand of $\mathfrak{B}_c$. This constitutes a generalization of the direct sum decomposition as it represents a decomposition with "minimal" intersection.

In this paper we denote the polynomial ring $k[x_1, x_2, \dots, x_n]$ of polynomials in n indeterminates with coefficients in the field $k = \mathbb{R}$ or $\mathbb{C}$, by $\mathfrak{D}$.

We mainly investigate these problems for $n = 2$ (i.e. $dim(\mathfrak{D}) = 2$), even though some results are still valid for any n.

II. Multidimensional Behaviors

In this section we review some concepts of nD behavioral systems. For a nice overview we refer to, for example, [9], [22] or [19].

In the behavioral approach to nD systems, a system is defined by a triple $(\mathcal{A}, q, \mathfrak{B})$, where $\mathcal{A}$ is the signal space, $q \in \mathbb{Z}^+$ is the number of components and $\mathfrak{B} \subset \mathcal{A}^q$ is the behavior. Here, we consider $\mathcal{A}$ the space of all infinitely often differentiable functions from $\mathbb{R}^n$ to k (denoted by $\mathfrak{C}^\infty(\mathbb{R}^n, k)$) or all k-valued distributions on $\mathbb{R}^n$ (denoted by $\mathcal{D}'(\mathbb{R}^n, k)$). The results of the paper are perfectly valid also for the discrete case $\mathcal{A} = k^{\mathbb{N}n}$. For the sake of simplicity we will however focus on the continuous case $\mathcal{A} = \mathfrak{C}^\infty(\mathbb{R}^n, k)$.

We call $\mathfrak{B}$ a *linear differential nD behavior* or simply *nD behavior* if it is the solution set of a system of linear, constant-coefficient partial differential equations, more precisely, if $\mathfrak{B}$ is the subset of $\mathcal{A}^q$ consisting of all solutions to

$$R(\frac{d}{d\mathbf{x}})w = 0 \tag{1}$$

where R is a polynomial matrix in n indeterminates x_i, $i = 1, \dots, n$, and $\frac{d}{d\mathbf{x}} = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. The elements of $\mathfrak{B}$ are called trajectories. We call (1) a *kernel representation* of $\mathfrak{B}$ and we write $\mathfrak{B} = \ker(R)$. Obviously, any linear differential nD behavior $\mathfrak{B}$ is a linear subspace of $\mathcal{A}^q$. Furthermore, it has the structure of a module over the ring of differential or difference operators.

It was shown in [7] that there is a one-to-one correspondence between nD behaviors and submodules of $\mathfrak{D}^q$. With any nD behavior $\mathfrak{B} \subset \mathcal{A}^q$ we associate the submodule $\mathfrak{B}^\perp$ of $\mathfrak{D}^q$ defined by

$$\mathfrak{B}^\perp := \{r \in \mathfrak{D}^q \mid r(\frac{d}{d\mathbf{x}})w = 0 \text{ for all } w \in \mathfrak{B}\}.$$

Conversely, for any submodule $\mathcal{M}$ of $\mathfrak{D}^q$ we have that

$$\mathcal{M}^\perp := \{w \in \mathcal{A}^q \mid r(\frac{d}{d\mathbf{x}})w = 0 \text{ for all } r \in \mathcal{M}\}$$

is an nD behavior. Indeed one has that $\mathfrak{B}^{\perp\perp} = \mathfrak{B}$ and $\mathcal{M}^{\perp\perp} = \mathcal{M}$. With this bijection, we have $(\mathfrak{B}_1 \cap \mathfrak{B}_2)^\perp = \mathfrak{B}_1^\perp + \mathfrak{B}_2^\perp$ and $(\mathcal{M}_1 \cap \mathcal{M}_2)^\perp = \mathcal{M}_1^\perp + \mathcal{M}_2^\perp$. If $\mathfrak{B} = \ker(R)$ then $\mathfrak{B}^\perp$ is the submodule of $\mathfrak{D}^q$ of all $\mathfrak{D}$-linear combinations of the rows of R.

The relation between $\mathfrak{B}$ and $\mathfrak{B}^\perp$ has provided many results and some authors reffer to it as a "duality". Even if it is strongly related it is not the same as the duality due to Malgrange [6] and defined as follows:

Let $Hom_{\mathfrak{D}}(\mathcal{M}, \mathcal{A}) = \{\mathfrak{D} - \text{linear map from } \mathcal{M} \rightarrow \mathcal{A}\}$ and $\mathfrak{B}$ any nD behavior then one has that $\mathfrak{B} = Hom_{\mathfrak{D}}(\mathfrak{D}^q/\mathfrak{B}^\perp, \mathcal{A})$.

Let $\mathcal{M}$ be a finitely generated $\mathfrak{D}$-module, we use the notation $D(\mathcal{M}) := Hom_{\mathfrak{D}}(\mathcal{M}, \mathcal{A})$ and $\mathcal{M}^* := Hom_{\mathfrak{D}}(\mathcal{M}, \mathfrak{D})$. We will omit an explicit reference to the ring $\mathfrak{D}$ as there will be no ambiguity and write $Hom(,)$ instead of $Hom_{\mathfrak{D}}(,)$.

We now introduce some basic definitions, mathematical tools and known results which will be needed in the rest of the paper.

Given the $\mathfrak{D}$-modules B, C, and E and the $\mathfrak{D}$-linear maps $\alpha : B \longrightarrow C$, we define

$$Hom(\alpha, E) : Hom(C, E) \longrightarrow Hom(B, E)$$
$$by \quad \varphi \longmapsto \varphi \circ \alpha.$$

Definition 2.1: A sequence $\cdots \longrightarrow A_{j-1} \xrightarrow{d_j} A_j \xrightarrow{d_{j+1}} A_{j+1} \longrightarrow \cdots$ of R-modules and homomorphisms d_j is called *exact* if for every (relevant) j one has $ker\ (d_{j+1}) = im\ (d_j)$ and is called *complex* if for every (relevant) j one has $ker\ (d_{j+1}) \supset im\ (d_j)$.
For example the sequence $0 \longrightarrow A_1 \xrightarrow{\alpha} A_2 \xrightarrow{\beta} A_3 \longrightarrow 0$ is exact if and only if α is injective, β surjective and $ker\ \beta = im\ \alpha$. In other words, A_1 can be identified with a submodule of A_2, and A_3 with the module of A_2/A_1. Exact sequences are an easy way to express algebraic and system-theoretic properties.
Oberst in [7] extended the work of Malgrange and Palamodov and proved the following fundamental theorem.

Theorem 2.1: Given finitely generated $\mathfrak{D}$-modules B, C and D, $\mathfrak{D}$-linear maps α and β, the complex

$$0 \longrightarrow B \xrightarrow{\alpha} C \xrightarrow{\beta} D \longrightarrow 0 \tag{2}$$

and its dual complex

$$0 \leftarrow Hom(B, \mathcal{A}) \xleftarrow{\tilde{\alpha}} Hom(C, \mathcal{A}) \xleftarrow{\tilde{\beta}} Hom(D, \mathcal{A}) \leftarrow 0 \tag{3}$$

then we have that (2) is exact if and only if (3) is exact.
The last theorem amounts to say that the signal space $\mathcal{A}$ is an injective cogenerator and is important to note that many

other signal spaces, e.g. the set of smooth functions with compact support, are not injective cogenerators [12].

Given a $\mathfrak{D}-$module $\mathcal{M}$, an element $m \in \mathcal{M}$ is called a *torsion element* if there exists $0 \neq d \in \mathfrak{D}$ such that $dm = 0$. The set of torsion elements is a submodule of $\mathcal{M}$. If this submodule is the 0-module, then $\mathcal{M}$ is called *torsion-free*.

In the behavioral approach, interconnection of systems is defined by intersection of the corresponding behaviors. Thus the interconnected behavior consists of the trajectories satisfying the equations of both systems, i.e. if $\mathfrak{B}_1$ =ker(R_1) and $\mathfrak{B}_2$ =ker(R_2) then $\mathfrak{B}_1 \cap \mathfrak{B}_2$ =ker$\left(\begin{smallmatrix} R_1 \\ R_2 \end{smallmatrix} \right)$.

Definition 2.2: The interconnection $\mathfrak{B}_1 \cap \mathfrak{B}_2$ is called *regular* if $\mathfrak{B}_1^{\perp} \cap \mathfrak{B}_2^{\perp} = 0$.
Hence we have that regular interconnection expresses the idea that the controller imposes new constrains on the plant which are not already present, i.e. there is no redundancy between the laws of the plant and the controller. Hence, feedback interconnections are regular interconnections since the controller imposes restrictions only on the input of the plant, which is unconstrained.

Definition 2.3: Let $\mathfrak{B} = \{w \in \mathcal{A}^q \mid R(\frac{d}{d\mathbf{x}})w = 0\} \subset \mathcal{A}^q$. Then the i-th component w_i of w is called *free* or *input* if $\pi_i : \mathfrak{B} \longrightarrow \mathcal{A}$ given by $w \longmapsto w_i$ is surjective. The behavior is called *autonomous* if has no free variables.

Theorem 2.2: (see [19], [21], [9]) Given $\mathfrak{B} = \{w \in \mathcal{A}^q \mid R(\frac{d}{d\mathbf{x}})w = 0\} \subset \mathcal{A}^q$. The following are equivalent:

1) $\mathfrak{B}$ is autonomous;
2) $\mathfrak{D}^q/\mathfrak{B}^{\perp}$ is torsion.

A strong form of autonomy is studied next.

Definition 2.4: A behavior $\mathfrak{B}$ is said to be *strongly autonomous* if it is finite dimensional as a vector space over k..

Theorem 2.3: (see [9], [8]) Given a behavior $\mathfrak{B} \subset \mathcal{A}^q$, the following are equivalent:

1) $\mathfrak{B}$ is strongly autonomous;
2) For every open non empty $U \subset \mathbb{R}^n$ the restriction map $r_U : \mathcal{A}^q \to \mathcal{A}|_U^q$ is injective on $\mathcal{B}$.

Thus, every trajectory of a strongly autonomous behavior is determined by its values on any open subset of $\mathbb{R}^n$.

Definition 2.5: A behavior $\mathfrak{B}$ is said to be *controllable* if for all w_1, $w_2 \in \mathfrak{B}$ and all sets U_1 ,$U_2 \subset \mathbb{R}^n$ with disjoint closure, there exist a $w \in \mathfrak{B}$ such that $w\mid_{U_1} = w_1\mid_{U_1}$ and $w\mid_{U_2} = w_2\mid_{U_2}$.

Theorem 2.4: (see [19], [21], [9]) Given a behavior $\mathfrak{B} \subset \mathcal{A}^q$, the following conditions are equivalent:

1) $\mathfrak{B}$ is controllable;
2) $\mathfrak{D}^q/\mathfrak{B}^{\perp}$ is torsion free.

The following definition was first introduced in [11], see also [10].

Definition 2.6: A behavior $\mathfrak{B} = D(\mathcal{M})$ is said to be *strongly controllable* if $\mathcal{M}$ is free.

Definition 2.7: A behavior $\mathfrak{B}$ is said to be *regular* if it has a full row rank kernel representation.

For the case $n = 1$, all behaviors are regular. This is not longer true for $n \geq 2$, take for instance the 2D differential behavior $\mathfrak{B}$ =ker$\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix} \right)$, consisting of all constant functions, which cannot be described as the kernel of a single polynomial operator.

III. A USEFUL THEOREM

In this section we will provide most of the technical results of the paper. The main algebraic tool we will use is *localization*. Localization is a systematic method of adding multiplicative inverses to a ring in order to construct local rings out of a ring. This notion allows us to reduce many questions concerning arbitrary rings to local rings. A ring is called *local* if it has exactly one maximal ideal. The unique maximal ideal consists precisely of the non-invertible elements of the ring.

Let R be a ring (always commutative with identity element 1), and $S \subset R$ a multiplicative set (i.e. $1 \in S$ and $s_1, s_2 \in S$ implies $s_1 s_2 \in S$). We introduce the following equivalence relation $\sim$ on $R \times S$:

$$(a, s) \sim (b, t) \Longleftrightarrow \exists u \in S \text{ such that } u(at - bs) = 0.$$

We will write a/s for the class of (a, s). Then the *ring of fractions* of R with respect to S, denoted by $S^{-1}R$, is $(R \times S)/\sim$ with ring operations defined by the usual arithmetic operations on fractions:

$$\frac{a}{s} \pm \frac{b}{t} = \frac{at \pm bs}{st} \qquad \text{and} \qquad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

Proposition 3.1: The following statements hold:

1) The ring operations are well defined, and $S^{-1}R$ is a ring.
2) $\varphi : R \longrightarrow S^{-1}R$ defined by $a \longmapsto a/1$ is a ring homomorphism.

Given a ring R, there are two popular and useful choices of multiplicative sets $S \subset R$:

1) $S = \{1, z, z^2, z^3, ...\}$, for a given element $z \in R$ and we write $R_z := S^{-1}R$.
2) $S = R \setminus \mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of the ring R. Then $S^{-1}R$ is a local ring with unique maximal ideal $\mathfrak{m} \cdot S^{-1}R$. The local ring $S^{-1}R$ is called the *localization* of R at $\mathfrak{m}$, and denoted by $R_{\mathfrak{m}} := S^{-1}R$.

Note that if $0 \in S$ then $S^{-1}R = 0$. We now proceed in a similar way with modules instead of ideals. Let M be an R-module and $S \subset R$ a multiplicative set. Define the equivalence relation $\sim$ on $M \times S$ as follows:

$$(m, s) \sim (n, t) \Longleftrightarrow \exists u \in S \text{ such that } u(tm - sn) = 0.$$

Denote $(M \times S)/\sim$ by $S^{-1}M$. This is again a module, this time over the ring $S^{-1}R$, with operations defined by:

$$\frac{m}{s} \pm \frac{m}{t} = \frac{tm \pm sn}{st} \qquad \text{and} \qquad \frac{a}{s} \cdot \frac{n}{t} = \frac{an}{st}.$$

If $S = R \setminus \mathfrak{m}$, with $\mathfrak{m}$ a maximal ideal, then $S^{-1}M$ is a module over the local ring $S^{-1}R = R_{\mathfrak{m}}$. This $S^{-1}R$-module is denoted by $M_{\mathfrak{m}}$. If $S = \{1, z, z^2, \dots\}$ for a given element $z \in R$ then we denote $S^{-1}M = M_z$.

Let $\mathfrak{D} = k[x_1, x_2, \dots, x_n]$, M be a finitely generated $\mathfrak{D}$-module and $S = \mathfrak{D} - \{0\}$. Hence $S^{-1}\mathfrak{D}$, denoted by $\mathfrak{D}_S$, is the set of rational fuctions and $S^{-1}M = \{\frac{m}{p} \mid m \in M,\ 0 \neq$

$p \in \mathfrak{D}\}$ is a vector space over the field $S^{-1}\mathfrak{D}$ and is denoted by M_S.

We will now introduce the notion of *codimension*. Codimension is a term used to indicate the difference between the dimension of certain objects and the dimension of a smaller object contained in it.

Definition 3.1: Let $A \subset B$ be finitely generated $\mathfrak{D}$-modules. A has *finite codimension* in B means that the dimension of B/A as a vector space over k is finite i.e. $dim_k(B/A) < \infty$, and is denoted by $A \underset{<\infty}{\subset} B$.

Lemma 3.1: Let $\mathfrak{D} = k[x_1, x_2, \dots x_n]$ and $\mathcal{M}$ be a finitely generated $\mathfrak{D}$-module. The following are equivalent:

1) $dim_k(\mathcal{M}) < \infty$;
2) For all x_i there exists a non constant $p_i(x_i) \in \mathfrak{D}[x_i]$ with $p(x_i)\mathcal{M} = 0,\ i = 1, \dots n$;
3) There exists an ideal $I \underset{<\infty}{\subset} \mathfrak{D}$ with $I \cdot \mathcal{M} = 0$;
4) $dim_k(Hom(\mathcal{M}, \mathcal{A})) < \infty$.

Furthermore, given $\mathfrak{B}_1, \mathfrak{B}_2$ two behaviors then $\mathfrak{B}_2 \underset{<\infty}{\subset} \mathfrak{B}_1 \Leftrightarrow \mathfrak{B}_1^\perp \underset{<\infty}{\subset} \mathfrak{B}_2^\perp$.

Proof : Any finitely generated $\mathfrak{D}$-module can be written in the form $\mathfrak{D}^q/N$ for some q and some submodule N of $\mathfrak{D}^q$.

(1) $\Rightarrow$ (2): Let $\{f_1, f_2, \dots f_s\}$ be a set of generators of $\mathcal{M} = \mathfrak{D}^q/N$. Consider the sequence $\{f_j, x_i f_j, x_i^2 f_j \dots\}$, since $dim_k(\mathcal{M}) < \infty$ one has that there exists $t \in \mathbb{N}$ such that $x_i^t f_j = a_{t-1} f_j x_i^{t-1} + a_{t-2} f_j x_i^{t-2} + \dots + a_0 f_j$, with $a_0, a_1, a_2, \dots, a_{t-1} \in k$. Define $p_{j,i}(x_i) := -x_i^t + a_{t-1}x_i^{t-1} + a_{t-2}x_i^{t-2} + \dots + a_0$ and clearly $p_{j,i}(x_i)\bar{f}_j = 0$. Hence one may compute $p_{i,j}$ for all $j = 1, 2, \dots s$, and $p_i(x_i) := p_{1,i}(x_i)p_{2,i}(x_i) \dots p_{s,i}(x_i)$ satisfies $p_i(x_i)\mathcal{M} = 0$. This holds for any $i = 1, \dots, n$.

(2) $\Rightarrow$ (3): Let t_i be the degree of the p_i. The basis of $\mathfrak{D}^p / < p_1, p_2, \dots, p_s >$ is $\{1, x_1, \dots, x_1^{t_1 - 1}, x_2, \dots x_2^{t_2-1}, \dots x_n^{t_n-1}\}$. Thus $I :=< p_1, p_2, \dots, p_s > \underset{<\infty}{\subset} \mathfrak{D}$ and $I \cdot \mathcal{M} = 0$.

(3) $\Rightarrow$ (4): Since $I \underset{<\infty}{\subset} \mathfrak{D}$ it is clear that there exists non constant $p_i(x_i) \in I,\ i = 1, \dots, n$. Let $\{f_1, f_2, \dots f_s\}$ be a set of generators of $\mathcal{M} = \mathfrak{D}^q/N$. Every $\ell \in Hom(\mathcal{M}, \mathcal{A})$ can be identified with $\ell(f_1), \ell(f_2), \dots, \ell(f_s) \in \mathcal{A}$. Hence it is enough to show that $\{\ \ell(f_1) \in \mathcal{A} \mid \ell \in Hom(\mathcal{M}, \mathcal{A})\}$ is finite dimension. Since $If_1 = 0 \Rightarrow I\ell(f_1) = 0$ one has that $\{\ \ell(f_1) \in \mathcal{A} \mid \ell \in Hom(\mathcal{M}, \mathcal{A})\} \subset \{g \in \mathcal{A} \mid Ig = 0\} \subset \{g \in \mathcal{A} \mid p_i(x_i)g = 0,\ i = 1, 2 \dots, n\}$. Thus it is enough to show that $\{g \in \mathcal{A} \mid p_i(x_i)g = 0,\ i = 1, 2 \dots, n\}$ is finite dimensional. This follows from the fact that $\{g \in \mathcal{A} \mid p_i(x_i)g = 0\ i = 1, 2 \dots, n\} = span_k\{x^{\alpha_i} exp(\lambda_i) \in \mathcal{A} \mid \lambda_i$ a root of p_i , $i = 1, 2 \dots, n,\ \alpha_i = 0, 1, \dots deg(\lambda_i) - 1\}$ which has finite dimension over k.

(4) $\Leftrightarrow$ (1): Already done in [8].

Finally, by theorem 2.1, one has that the exact sequence

$$0 \longrightarrow \mathfrak{B}_2^\perp/\mathfrak{B}_1^\perp \longrightarrow \mathfrak{D}^q/\mathfrak{B}_1^\perp \longrightarrow \mathfrak{D}^q/\mathfrak{B}_2^\perp \longrightarrow 0 \qquad (4)$$

implies that

$$0 \longleftarrow Hom(\mathfrak{B}_2^\perp/\mathfrak{B}_1^\perp, \mathcal{A}) \longleftarrow \mathfrak{B}_1 \longleftarrow \mathfrak{B}_2 \longleftarrow 0 \qquad (5)$$

is exact which means that $Hom(\mathfrak{B}_2^\perp/\mathfrak{B}_1^\perp, \mathcal{A}) \approx \mathfrak{B}_1/\mathfrak{B}_2$. Hence, $\mathfrak{B}_2^\perp/\mathfrak{B}_1^\perp$ is finite dimension if and only if $Hom(\mathfrak{B}_2^\perp/\mathfrak{B}_1^\perp, \mathcal{A}) \approx \mathfrak{B}_1/\mathfrak{B}_2$ is finite dimension using the equivalence of the statements (1) and (4). ☐

Throughout the rest of this section we will take $n = 2$ and consider the ring $\mathfrak{D} = k[x_1, x_2]$. The results we will obtain are valid for this particular ring.

Lemma 3.2: Let $\mathfrak{D} = k[x_1, x_2]$, $I \subset \mathfrak{D}$ be an ideal and $\{f_1, ...f_m\}$ be a generating set for I. Then the following three statements are equivalent:

1) $I \underset{<\infty}{\subset} \mathfrak{D}$;
2) the greatest common divisor (g.c.d) of $\{f_1, ...f_m\}$ is equal to 1;
3) $Z(I) = \{(a_1, a_2) \in \mathbb{C}^2 \mid \forall f \in I\ f(a_1, a_2) = 0\}$ is finite.

Proof : First recall that radical$(I) = \sqrt{I} = \{d \in \mathfrak{D} \mid d^m \in I$ for some positive integer $m\}$.

(1) $\Rightarrow$ (2): Suppose that $g.c.d(f_1, \dots f_m) = f$ with f not constant, clearly $I \subset (f) \subset \mathfrak{D}$. A consequence of the Noether normalization is that after a linear change of variables f can be written as $f = x_2^d + a_{d-1}(x_1)x_2^{d-1} + \dots a_0(x_1)$ and therefore $\mathfrak{D}/(f)$ is a free $k[x_1]$-module of rank d with infinite dimension over k.

(2) $\Rightarrow$ (3): Every f_i can be decomposed as $f_i = g_1 g_2 \dots g_r$ with g_j irreducible polynomials and $\sqrt{(g_j)}$ corresponds to an irreducible curve in $\mathbb{C}^2$ (i.e. $Z(\sqrt{(g_i)}$ is an irreeducible curve). Hence $\sqrt{(f_i)}$ corresponds to a curve Γ_i which is a finite union of irreducible curves, and therefore $Z(\sqrt{I})$ corresponds to the intersection of all Γ_i. The assumption $g.c.d(\{f_1, ...f_m\}) = 1$ means that the curves Γ_i do not coincide anywhere and therefore they intersect just in points , i.e. the set $Z(\sqrt{I}) = Z(I)$ contains just points and it is finite because $\Gamma_1, \Gamma_2, \dots, \Gamma_r$ can not intersect infinite many times.

(3) $\Rightarrow$ (1): Since $Z(I) = Z(\sqrt{(I)}) \Rightarrow Z(\sqrt{(I)})$ is finite. By definition $\sqrt{I} = \cap_{P_i \supset I} P_i$, P_i prime ideals. For all $P_i \supset I$ one has that $Z(\sqrt{I}) \supset Z(P_i)$ which implies P_i is a maximal ideal (since $Z(\sqrt{I})$ finite). For any two maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2 \subset \mathfrak{D}$ one has that $Z(\mathfrak{m}_1) = Z(\mathfrak{m}_2) \Leftrightarrow \mathfrak{m}_1 = \mathfrak{m}_2$, so there exists finite number of P_i such that $\sqrt{I} = \cap_{P_i \supset I} P_i$. Conclusion, $\sqrt{I} = \cap_{i=1}^t \mathfrak{m}_i$ for some $t \in \mathbb{N}$. Each maximal ideal is of the form $\mathfrak{m} = (p, q) \subset \mathfrak{D}$ where $p \in \mathbb{R}[x_1]$ irreducible and $q \in \mathfrak{D}$ irreducible in $(\mathbb{R}[x_1]/(p))[x_2]$. We claim that there exists $h(x_1, x_2) = x_2^r + x_2^{r-1}q_{r-1}(x_1) + \dots q_0(x_1) \in \mathfrak{m}_i$ for some $q_{r-1}, \dots, q_0 \in \mathbb{R}[x_1]$. Proof of the claim: $g_i(x_1, x_2) \in \mathbb{R}[x_1][x_2]$, with not all coefficients in (p_i) (otherwise $\mathfrak{m}_i = (p_i)$ which is not maximal). One can always write $g_i(x_1, x_2) = x_2^s q_s(x_1) + x_2^{s-1}q_{s-1}(x_1) + \dots + q_0(x_1)$. Take $h_i \equiv g_i(x_1, x_2) \mod(p),\ h_i \neq 0$. Since $(\mathbb{R}[x_1]/(p_i))$ is a field there exists $f \in \mathbb{R}[x_1]$ such that $fh_i \equiv x_2^{s'} + x_2^{s'-1}q_{s'-1}(x_1) + \dots + q_0(x_1) \mod(p_i) \Rightarrow x_2^{s'} + x_2^{s'-1}q_{s'-1}(x_1) + \dots + q_0(x_1) \in \mathfrak{m}_i$. Finally $p = p_1 p_2 \cdots p_t = x_1^{N_1}$+lower order terms $\in \mathbb{R}[x_1]$ and $h = h_1 h_2 \cdots h_t = x_2^{N_2}$+ (lower order terms in x_2) $\in \mathbb{R}[x_1][x_2]$. There exists $N \in \mathbb{N}$ such that $p^N, h^N \in I \Rightarrow \mathfrak{D}/I$ has finite dimension over k.

□

Lemma 3.3: Let $\mathfrak{m}$ be a maximal ideal of $\mathfrak{D}$. Let $\mathfrak{D}_\mathfrak{m}$ be the localized local ring of $\mathfrak{D}$ at $\mathfrak{m}$, and let z_1, z_2 be generators of the unique maximal ideal in $\mathfrak{D}_\mathfrak{m}$. Suppose that a finitely generated $\mathfrak{D}_\mathfrak{m}$-module N has the properties: N has no torsion and $N = N_{z_1} \cap N_{z_2}$. Then N is free.

Proof : We *claim* that the $\mathfrak{D}_\mathfrak{m}/(z_2)$-module $\overline{N} := N/z_2N$ has no torsion. Indeed, if this module has torsion then there exists a non zero element $\overline{n} \in \overline{N}$, image of $n \in N$, with $a\overline{n} = 0$ with $a \in \mathfrak{D}_\mathfrak{m}/(z_2)$. Since $a = cz_1$ with c invertible, then we may take $a = z_1$. Thus $z_1n \in x_2N$ and we write $z_1n = z_2\tilde{n}$ and $\xi := \frac{1}{z_2}n = \frac{1}{z_1}\tilde{n}$. Then $\xi \in N_{z_1} \cap N_{z_2} = N$ and $n = z_2\xi$ is in contradiction with $\overline{n} \neq 0$.

Since $\mathfrak{D}_\mathfrak{m}/z_2\mathfrak{D}_\mathfrak{m}$ is a principal ideal domain (moreover is a discrete valuation ring, see [1]) every torsion free $(\mathfrak{D}_\mathfrak{m}/z_2\mathfrak{D}_\mathfrak{m})$-module is free. Hence the module $\overline{N}$ is free and we can choose elements $n_1, \ldots, n_s \in N$ such that their images in $\overline{n}_1, \ldots, \overline{n}_s$ in $\overline{N}$ form a free basis. In particular, their images $\{\overline{\overline{n_i}}\}$in $\overline{\overline{N}} = N/(x_1, x_2)N$ form a basis over the residue field of $\mathfrak{D}_\mathfrak{m}$. It follows (by Nakayama's lemma) that the $\{n_i\}$ generate N. Suppose now that there is a non trivial relation $f_1n_1 + \cdots + f_sn_s = 0$. Then all f_i lie in the maximal ideal of $\mathfrak{D}_\mathfrak{m}$. Since N has no zero divisors, one may divide by the g.c.d. of all f_i and find a relation, again written as $f_1n_1 + \cdots + f_sn_s$, where the g.c.d. of all f_i is 1.

Write $f_i = z_1g_i + z_2h_i$ with $g_i, h_i \in \mathfrak{D}_\mathfrak{m}$. Then $z_1(\sum g_in_i) + z_2(\sum h_in_i) = 0$. Thus $z_1(\sum g_in_i) \in z_2N$ and, since $\overline{N}$ has no torsion, $\sum g_in_i$ has image 0 in $\overline{N}$. Since $\{\overline{n}_i\}$ are free, one finds that all $g_i \in z_2\mathfrak{D}_\mathfrak{m}$. The latter leads to the contradiction that all f_i are divisible by z_2. □

The following theorem and its corollaries will be essential for the rest of the paper and will be used in most of the results. First, we will recall a lemma (see ...) that will allow us to check the freeness of a module over an arbitrary ring by checking the freeness of certain modules over a local ring.

Lemma 3.4: ([1]) Let R be a ring (commutative with 1, the identity element). Let N be an R-module. Then N is projective if and only if for all maximal ideals $\mathfrak{m}$ of R, $N_\mathfrak{m}$ is free over $R_\mathfrak{m}$.

Now we present the main technical result of this paper:

Theorem 3.1: Let M be a finitely generated torsion free $\mathfrak{D}$-module. Then there exists a free $\mathfrak{D}$-module N such that $M \underset{<\infty}{\subset} N$.

Proof : M is contained in some free module F (see [5] p.44). Let $\{f_1, \ldots f_m\}$ be a basis of F and $m \in F_{x_1} \cap F_{x_2}$. One can write $m = a_1f_1 + \ldots a_mf_m = b_1f_1 + \ldots b_mf_m$ where $a_i \in \mathfrak{D}_{x_1}, b_i \in \mathfrak{D}_{x_2}$, and therefore $a_if_i = b_if_i \Rightarrow (a_i - b_i)f_i = 0 \Rightarrow a_i - b_i = 0 \Rightarrow a_i, b_i \in \mathfrak{D}_{x_1} \cap \mathfrak{D}_{x_2} \Rightarrow a_i, b_i \in \mathfrak{D}$, $i = 1, \ldots m$. Hence $F_{x_1} \cap F_{x_2} = F$ and $N := M_{x_1} \cap M_{x_2} \subset F_{x_1} \cap F_{x_2} = F$ is a finitely generate $\mathfrak{D}$-module containing M.

$\mathfrak{D}_\mathfrak{m}$ is a regular local ring of dimension two and we show that the $\mathfrak{D}_\mathfrak{m}$-module $N_\mathfrak{m}$ satisfies the requirements of the lemma 3.3 and thus N is free by lemma 3.4.

Finally, there is an integer $a \geq 1$ such that $(x_1, x_2)^aN/M = 0$. This implies that N/M has finite dimension over k. □

As an immediately consequence we obtain the following result for 2D behaviors:

Corollary 3.1: Let $\mathfrak{B}$ =ker$(R) = D(\mathcal{M})$ be a $2D$ behavior. There exists a regular behavior $\mathfrak{B}'$ such that $\mathfrak{B}' \underset{<\infty}{\subset} \mathfrak{B}$. Moreover if $\mathfrak{B}$ is controllable then there exists a strongly controllable $2D$ behavior $\mathfrak{B}'$ such that $\mathfrak{B} = \mathfrak{B}'/\tilde{\mathfrak{B}}$ with $\tilde{\mathfrak{B}}$ finite dimension over k.

Proof : The $\mathfrak{D}-$module $\mathfrak{B}^\perp \subset \mathfrak{D}^q$ is a torsion free module. By theorem 3.1 there exists a free $\mathfrak{D}$-module F such that $\mathfrak{B}^\perp \underset{<\infty}{\subset} F$ and $\mathfrak{B} \underset{<\infty}{\subset} F^\perp = \mathfrak{B}'$ by lemma 3.1.

If the $\mathfrak{D}$-module $\mathcal{M}$ is a torsion free module by theorem 3.1 there exists a free $\mathfrak{D}$-module F such that $\mathcal{M} \underset{<\infty}{\subset} F$. Obviously $\mathfrak{B}' = D(F)$ is strongly controllable since F is free. Consider the following exact sequence:

$$0 \longrightarrow \mathcal{M} \longrightarrow F \longrightarrow F/\mathcal{M} \longrightarrow 0 \tag{6}$$

Using the injective and cogenerator properties of $\mathcal{A}$, we have

$$0 \longrightarrow Hom(F/\mathcal{M}, \mathcal{A}) =: \tilde{\mathfrak{B}} \longrightarrow \mathfrak{B}' \longrightarrow \mathfrak{B} \longrightarrow 0 \tag{7}$$

is exact. Thus $\mathfrak{B} = \mathfrak{B}'/\tilde{\mathfrak{B}}$ and $\tilde{\mathfrak{B}}$ is finite dimension over k since $F/\mathcal{M}$ is finite dimension over k. □

We make some observations in the following lemma:

Lemma 3.5: Let M be a finitely generated torsion free $\mathfrak{D}$-module, N a free $\mathfrak{D}$-module such that $M \underset{<\infty}{\subset} N$ and $S = \mathfrak{D} - \{0\}$. The following holds:

1) $N_S = M_S$
2) If $\xi \in N$ then the ideal $I = \{p \in \mathfrak{D} \mid p\xi \in M\} \underset{<\infty}{\subset} \mathfrak{D}$ and therefore there is an ideal I with finite codimension such that $I \cdot N \subset M$.
3) If $\xi \in N_S = M_S$, but does not belong to N, then the ideal $I = \{f \in \mathfrak{D} \mid f\xi \in N\}$ (and hence also the ideal $J = \{f \in \mathfrak{D} \mid f\xi \in M\}$) does not have finite codimension in $\mathfrak{D}$.

Proof : Note that such an N always exists from lemma 3.1.

(1) : There exists $d \in \mathfrak{D}$ such that $dn \in M$ for all $n \in N$ since $M \underset{<\infty}{\subset} N$. Hence for all $n' \in N_S \Rightarrow dn' \in M_S \Rightarrow n' = dn'/d \in M_S \Rightarrow M_S \supset N_S$ and $M_S \subset N_S$ is obvious.

(2) : Follows from statement (3) of lemma 3.1.

(3) : Write $\{e_1, ..., e_m\}$ for a free basis of N and $\xi = \xi_1e_1 + \cdots + \xi_me_m$ with all $\xi_i \in \mathfrak{D}_S$. Using that $\mathfrak{D}$ is a unique factorization domain one can write each ξ_i as $f_i/g_i,\ f_i, g_i \in \mathfrak{D}$ with $g.c.d(f_i, g_i) = 1$. Let g be the smallest common multiple of $g_1, \ldots, g_m$. Thus $I = g\mathfrak{D} \subset \mathfrak{D}$ and does not have finite codimension. □

Corollary 3.2: Let M be a finitely generated torsion free $\mathfrak{D}$-module and $S = \mathfrak{D} - \{0\}$. Thus $N = \{\xi \in M_S \mid \{f \in$

$\mathfrak{D} \mid f\xi \in M\}$ has finite codimension $\}$ is the unique free $\mathfrak{D}$-module such that $M \underset{<\infty}{\subset} N$ and therefore if M is free then $N = M$. We write M^+ for N. Moreover, $(M^+)^+ = M^+$. The computation of M^+ will be essential for solving the problems we have considered. We now provide a theorem which allows to easily compute M^+.

Theorem 3.2: Let M be a finitely generated torsion free module over $\mathfrak{D}$ and M^+ the free $\mathfrak{D}$-module such that $M \underset{<\infty}{\subset} M^+$. Then $M^+ = M^{**}$, thus the natural map $M \to M^{**}$ is the required embedding.

Proof : The exact sequence $0 \to M \to F \to F/M \to 0$ induces the exact sequence $0 \to Hom(F/M, R) \to F^* \to M^* \to Ext^1(F/M, R) \to 0$. Now $Hom(F/M, R) = 0$ and $ext^1(F/M, R)$ has finite dimension. Thus $F^* \subset M^*$ and $\dim M^*/F^* < \infty$. Take a basis $e_1, \dots, e_d$ of F^* and consider an element $b \in M^*$. Write $b = f_1 e_1 + \cdots + f_d e_d$ with $f_1, \dots, f_d \in k(x_1, x_2)$. The ideal $I := \{f \in R \mid fb \in F^*\}$ is the principle ideal generated by the smallest common multiple of the denominators of $f_1, \dots, f_d$. Now $(F^* + Rb)/F^* \cong R/I$. Suppose that $I \neq R$, then $\dim R/I = \infty$ and this contradicts $\dim M^*/F^* < \infty$. We conclude that $I = R$ and $b \in F^*$. Thus $F^* = M^*$ and $F^{**} = M^{**}$. $\square$

IV. "ALMOST" REGULAR IMPLEMENTABILITY

We consider the following problem:

Problem 1: Given a behavior $\mathfrak{B}$ and a control objective $\mathfrak{B}_d \subset \mathfrak{B}$, find a behavior $\mathfrak{B}_c$ (a controller) such that

$$\mathfrak{B} \cap \mathfrak{B}_c \underset{<\infty}{\subset} \mathfrak{B}_d \text{ and the interconnection is regular.}$$

If such $\mathfrak{B}_c$ exists, then we say that $\mathfrak{B}_d$ is almost implementable by regular interconnection from $\mathfrak{B}$. The problem of finding such a $\mathfrak{B}_c$ is called the problem of almost regular implementability of $\mathfrak{B}_d$ from $\mathfrak{B}$.

In this section we aim to investigate under what conditions a given $\mathfrak{B}_d \subset \mathfrak{B}$ is almost implementable by regular interconnection from $\mathfrak{B}$.

The following theorem reduces this problem to the problem of checking whether a given free module is a direct summand of a larger free module.

Theorem 4.1: Given behaviors $\mathfrak{B}_d \subset \mathfrak{B}$, denote $N_1 = \mathfrak{B}^\perp$, $N = \mathfrak{B}_d^\perp$. If $\mathfrak{B}_d$ is almost implementable by regular interconnection from $\mathfrak{B}$ then N_1^+ is direct summand of N^+. Furthermore, if N_1^+ is direct summand of N^+ and $\mathfrak{B}$ is a regular behavior then $\mathfrak{B}_d$ is almost implementable by regular interconnection from $\mathfrak{B}$.

Proof : Using the relation between $\mathfrak{B}$ and $\mathfrak{B}^\perp$, and taking $N_1 = \mathfrak{B}^\perp$, $N = \mathfrak{B}_d^\perp$ the problem of almost regular implementability can be re-stated as follows: Given modules $N_1 \subset N \subset \mathfrak{D}^q$, does there exist a $\mathfrak{D}$-module $N_2 \subset \mathfrak{D}^q$ such that $N \underset{<\infty}{\subset} N_1 + N_2$ and $N_1 \cap N_2 = 0$.

Note that if $N_1 \subset N^+$ then $N_1^+ \subset (N^+)^+ = N^+$. Since $Hom(N_1 \oplus N_2, \mathfrak{D}) = Hom(N_1, \mathfrak{D}) \oplus Hom(N_2, \mathfrak{D})$ one has that $(N_1 \oplus N_2)^+ = N_1^+ \oplus N_2^+$.

(a) Suppose there exists N_2 such that $N \underset{<\infty}{\subset} N_1 \oplus N_2$, then $N^+ \underset{<\infty}{\subset} (N_1 \oplus N_2)^+ = N_1^+ \oplus N_2^+$. Using (3) of lemma 3.5 one has that $N_1^+ \oplus N_2^+ = N^+$.

(b) Suppose there exists N_2^+ such that $N_1^+ \oplus N_2^+ = N^+$ and N_1 is free (because $\mathfrak{B}$ is a regular behavior), i.e. $N_1^+ = N_1$, this implies $N \underset{<\infty}{\subset} N_1^+ \oplus N_2^+ = N_1 \oplus N_2^+$. $\square$

Theorem 4.1 provides a necesary and a sufficient condition for solving the problem of almost regular implementability of $\mathfrak{B}_d$ from $\mathfrak{B}$. Indeed, such a necessary condition (and also sufficient if N_1 is free) can be computed by checking whether N_1^+ is direct summand of N^+. This is computationally very effective since both N^+ and N_1^+ are free modules.

One way to do it is to construct a matrix R which maps a basis $e_1, e_2, \dots e_n$ of N_1^+ to the basis $f_1, f_2, \dots f_m$ of N^+.

If the $n \times n$ minors of R do not generate the unit ideal $\mathfrak{D}$, then $\mathfrak{B}_d$ is not almost achievable by regular interconnection from $\mathfrak{B}$.

If $n \times n$ minors of R generate the unit ideal $\mathfrak{D}$ and N_1 is free, then N_1 is direct summand of N and one can compute elements $\{e_{n+1}, \dots e_m\} \in N$ such that $\{e_{n+1}, \dots e_m\}$ is a basis of N_2^+. This means that N_2^+ can be computed explicitly and therefore the controller $\mathfrak{B}_c = (N_2^+)^\perp$.

V. AUTONOMOUS-CONTROLLABLE DECOMPOSITION WITH FINITE DIMENSIONAL INTERSECTION

In the next two sections we address the problem of decomposing a behavior into smaller components. We do it in two steps. First in this section we look at the autonomous-controllable decomposition and in the next section we treat the decomposition of the controllable part. The autonomous-controllable decomposition has played an important role in the theory of linear systems. It has been studied intensively in the context of $1D$ behaviors [16], in the context for $2D$ behaviors for [4], [15], and for higher dimensional in [19], [21]. In our search to decompose a given behavior into simpler components, it seems natural to study first weather it is possible to have an autonomous-controllable decomposition with finite dimensional intersection. In this section we show that autonomous-controllable decomposition with finite dimensional intersection is always feasible for 2D behaviors and provide a counterexample for $n = 3$.

Lemma 5.1: Let $\mathfrak{B}_{cont}$ be the controllable part of a given behavior $\mathfrak{B}$. Denote $M := \mathfrak{D}/\mathfrak{B}^\perp$ and let M_t be the torsion submodule of M. Construct the exact sequence $0 \longrightarrow M_t \longrightarrow M \xrightarrow{\beta} N \approx M/M_t \longrightarrow 0$. The following statements are equivalent:

1) There exists an autonomous behavior $\mathfrak{B}_{aut} \subset \mathfrak{B}$ such that
$$\mathfrak{B} = \mathfrak{B}_{cont} + \mathfrak{B}_{aut} \text{ and}$$
$$\mathfrak{B}_{cont} \cap \mathfrak{B}_{aut} \text{ has finite dimension;} \qquad (8)$$
2) There exists an autonomous behavior $\mathfrak{B}_{aut} \subset \mathfrak{B}$ such that
$$\mathfrak{B}^\perp = (\mathfrak{B}_{cont})^\perp \cap (\mathfrak{B}_{aut})^\perp \text{ and}$$

$$(\mathfrak{B}_{cont})^{\perp} + (\mathfrak{B}_{aut})^{\perp} \underset{<\infty}{\subset} \mathfrak{D}^q; \tag{9}$$

3) There exists a $\mathfrak{D}$-module $A \subset M$ such that $D(M) = D(M/M_t) + D(M/A)$ and $D(M/M_t) \cap D(M/A)$ has finite dimension;
4) There exists a $\mathfrak{D}$-module $A \subset M$ such that $A \cap M_t = 0$ and $A + M_t \underset{<\infty}{\subset} M$;
5) There exists a $\mathfrak{D}$-module $N' \underset{<\infty}{\subset} N$ such that $0 \longrightarrow M_t \longrightarrow M' := \beta^{-1}(N') \xrightarrow{\beta} N \longrightarrow 0$ splits, i.e. M_t is direct summand of M'.

The following remark and lemma will be needed in the proof of lemma 5.1.

Remark 2: Consider the following exact sequence:

$$0 \longrightarrow M_t \longrightarrow M \xrightarrow{\beta} N \longrightarrow 0 \tag{10}$$

We could replace N by a submodule N' with $dim(N/N') < \infty$ and M by $M' = \beta^{-1}(N')$ (and therefore $dim(M/M') < \infty$ since $\frac{N}{N'} \approx \frac{M/M_t}{M'/M_t} \approx \frac{M}{M'}$) and get the exact sequence

$$0 \longrightarrow M_t \longrightarrow M' \xrightarrow{\beta} N' \longrightarrow 0. \tag{11}$$

Lemma 5.2: Let $A \subset M$ be two $\mathfrak{D}$-modules and M_t the torsion part of M. Then the following hold:

(a) $M_t \cap A = 0 \Leftrightarrow D(M) = D(M/M_t) + D(M/A)$.

(b) $M_t + A \subset M \Leftrightarrow D(M/(M_t + A)) = D(M/M_t) \cap D(M/A)$.

Proof : Easy using corollary 3 in [18].

□

Proof of lemma 5.1: The equivalence of $(1), (2)$ and (3) follows straightfoward from the results provided in section II.

$(3) \Leftrightarrow (4)$:Since $M_t \cap A = 0 \Leftrightarrow D(M) = D(M/M_t) + D(M/A)$ is exactly part (a) of lemma 5.2, we just need to prove that $M_t + A \subset M$ with finite codimension $\Leftrightarrow$ $D(M/M_t) \cap D(M/A)$ has finite dimension. Using part (b) of lemma 5.2 one has that $D(M/M_t) \cap D(M/A) = D(M/(M_t+A))$, and $D(M/(M_t+A))$ has finite dimension if and only if $M/(M_t + A)$ has finite dimension, see [8], which means that $M_t + A \underset{<\infty}{\subset} M$.

$(4) \Leftarrow (5)$: Since the sequence splits there exists a $\mathfrak{D}$-module A such that $M_t \oplus A = M'$ and by remark 2 one has that $M' \underset{<\infty}{\subset} M$.

$(4) \Rightarrow (5)$: Let $N' := (M_t \oplus A)/M_t$. Then the sequence $0 \longrightarrow M_t \longrightarrow M_t \oplus A \xrightarrow{\beta} (M_t \oplus A)/M_t \longrightarrow 0$ is exact, splits and $(M_t \oplus A)/M_t \underset{<\infty}{\subset} (M/M_t$ since $M_t \oplus A \underset{<\infty}{\subset} M$. □

The following theorem state that the autonomous-controllable decomposition with finite dimensional intersection is always feasible.

Theorem 5.1: Let M be a finitely generated $\mathfrak{D}$-module and let M_t be the torsion submodule of M. Then there exists a submodule $A \subset M$ such that $A \cap M_t = 0$ and $A + M_t \subset M$ has finite codimension.

Proof : Using remark 2, it is enough to check that there exists $N' \subset N$ of finite dimension such that the sequence (11) splits. There exists $d \in R,\ d \neq 0$ with $dM_t = 0$. Since $N \cong M/M_t$, N is torsion free and applying theorem 3.1 there is a free module $F = Re_1 + \cdots + Re_m \supset N$ such that F/N has finite dimension. By lemma 3.1 there is an ideal $J \underset{<\infty}{\subset} \mathfrak{D}$ such that N contains the submodule $Je_1 + \cdots + Je_m$ of the free module $\mathfrak{D}e_1 + \cdots + \mathfrak{D}e_m$ i.e. $J \cdot F/N = 0$.

Further J contains a non zero multiple p of d since $J \underset{<\infty}{\subset} \mathfrak{D}$. We want to show that there exists an element $q \in J$ such that the ideal $(p, q) \underset{<\infty}{\subset} \mathfrak{D}$ (or equivalently g.c.d.$(p, q) = 1$).

The radical $\sqrt{J}$ corresponds to a finite set S of points in the plane $\mathbb{C}^2$. For any non zero element $f \in J$, the radical $\sqrt{(f)}$ corresponds to a curve Γ passing through S. This curve Γ is a finite union of irreducible curves. The converse is valid: let Γ be a curve, passing through S, then the radical ideal corresponding to Γ has the form Rg for some element $g \in \sqrt{J}$ and thus, for some integer $N \geq 1$ one has $g^N \in J$.

Now $\sqrt{Rp}$ defines a curve Γ passing through S, which is a finite union of irreducible curves $\Gamma_1, \ldots, \Gamma_r$. It is clear that there exists a curve Γ' passing through S, corresponding to some radical ideal Rg, such that none of the irreducible components of Γ' coincides with a Γ_i. Let $q := g^N \in J$. The radical ideal $\sqrt{(p, q)}$ corresponds to the intersection $\Gamma \cap \Gamma'$. This is a finite set and thus $\sqrt{(p, q)}$ and also $I := (p, q)$ are ideals in R with finite codimension. In particular g.c.d.$(p, q) = 1$.

We may replace N by $Ie_1 + \cdots + Ie_m$ and M accordingly. Choose, for $i = 1, \ldots, m$, elements $u_i, v_i \in M$ with images pe_i, qe_i in N.

Consider an expression $\sum_{i=1}^m (a_i u_i + b_i v_i)$ (all $a_i, b_i \in R$) having image 0 in N. Then $\sum_{i=1}^m (a_i p + b_i q)e_i = 0$ and it follows that for suitable $c_i \in R$ we have $a_i = c_i q,\ b_i = -c_i p$. Note that $qu_i - pv_i \in M_t$ for all i and thus $p(qu_i - pv_i) = 0$.

Next, consider the submodule A of M generated by the elements pu_i, qv_i for all i. Then $\beta(A) = (p^2, q^2)e_1 + \cdots + (p^2, q^2)e_m$ has finite codimension in N. We verify now that $A \cap M_t = 0$.

Suppose that $\xi := \sum_{i=1}^m (a_i p u_i + b_i q v_i)$ lies in M_t. Then there are c_i such that $a_i p = c_i q,\ b_i q = -c_i p$ for all i. Then c_i is divisible by pq since g.c.d.$(p, q) = 1$. Write $c_i = pqd_i$. Then $\xi = \sum_{i=1}^m pqd_i(qu_i - pv_i)$. This expression is zero. □

The following counterexample shows that the theorem 5.1 does not hold for $n \geq 2$.

Example 3: Take $\mathfrak{D} = k[x_1, x_2, x_3]$, and the exact sequence

$$0 \longrightarrow \mathfrak{D}(x_2, -x_1) \longrightarrow \mathfrak{D}^2 \xrightarrow{\beta} I \longrightarrow 0$$

with $\beta(a, b) = ax_1 + bx_2$, $I = (x_1, x_2) \subset \mathfrak{D}$. Furthermore,

$$0 \longrightarrow M_t \longrightarrow M \xrightarrow{\beta} I \longrightarrow 0$$

is exact, with $\mathfrak{D}(x_2, -x_1)/\mathfrak{D}x_1(x_1, -x_2) = M_t$ and $\mathfrak{D}^2/\mathfrak{D}x_1(x_1, -x_2) = M$. There does not exist $A \subset M$ with the required properties. In order to see this we take an ideal $J \subset I$ with finite codimension and check that the exact sequence $0 \longrightarrow M_t \longrightarrow \beta^{-1}(J) \xrightarrow{\beta} J \longrightarrow 0$ never splits. All ideals $J \subset I$ with finite codimension have the form $J = (x_1^s, x_2^r, x_1 x_3^t, x_2 x_3^w)$ for some $s, r, t, w \in \mathbb{Z}^+$. Then $\beta^{-1}(J) = (\beta^{-1}(x_1^s), \beta^{-1}(x_2^r), \beta^{-1}(x_1 x_3^t), \beta^{-1}(x_2 x_3^w)) = ((x_1^{s-1}, 0), (0, x_2 r - 1), (x_3^t, 0), (0, x_3^w))$. Take $m_1 = (x_2 x_3^{t+w}, 0), m_2 = (0, -x_1 x_3^{t+w} \in \beta^{-1}(J)$ but are not in M_t. Finally note that $m_1 + m_2 \in M_t$.

The next example shows that not every sub-behavior is an almost direct summand of a given behavior, not even for $n = 2$.

Example 4: Take $\mathfrak{D} = k[x_1, x_2]$, and the exact sequence

$$0 \longrightarrow \mathfrak{D}(x_2, -x_1) \longrightarrow \mathfrak{D}^2 \xrightarrow{\beta} I \longrightarrow 0$$

with $\beta(a, b) = ax_1 + bx_2$, $I = (x_1, x_2) \subset \mathfrak{D}$. There does not exist $A \subset M$with $A \cap \mathfrak{D}(x_2, -x_1)$ and $A + \mathfrak{D}(x_2, -x_1) \subset$ finite codimension. Following the idea in previous example, $J = (p(x_1)x_1, q(x_2)x_2)$ for any polynomials p, q. Then $\beta^{-1}(J) = ((p(x_1, 0)), (0, q(x_2)))$. Finally take $m_1 = (x_2 qp, 0), m_2 = (0, -x_1 pq)$.

VI. Decomposition of the controllable part

Once we have decomposed the behavior into autonomous and controllable parts, one may look for a finer decomposition. The autonomous part of $2D$ behaviors has been studied in detail in [15], [14], [2], [4]. Indeed, autonomous $2D$ behaviors, which are kernels of full column rank matrices, can be represented as a direct sum of two sub-behaviors. One finite dimensional and the other one a *square autonomous* behavior which is defined by a nonsingular square polynomial matrix. Hence, continuing our search for more elementary components, we look, in this section, at the controllable part of a given behavior. In fact, we treat the following problem:

Problem 5: Given a controllable $2D$ behavior $\mathfrak{B} \subset \mathcal{A}^q$ and a sub-behavior $\mathfrak{B}_a \subset \mathfrak{B}$ find a behavior $\mathfrak{B}_b \subset \mathfrak{B}$ such that

$$\begin{aligned} &1. \quad \mathfrak{B}_b + \mathfrak{B}_a = \mathfrak{B} \\ &2. \quad \mathfrak{B}_b \cap \mathfrak{B}_a \text{ has finite dimension .} \end{aligned} \tag{12}$$

Lemma 6.1: Let $\mathfrak{B}_a \subset \mathfrak{B} \subset \mathcal{A}$ be two $2D$ behaviors and denote $M = \mathfrak{D}^q / \mathfrak{B}^\perp$ and $N_1 = \mathfrak{B}_a^\perp / \mathfrak{B}^\perp$. The following problems are equivalent:

1) Problem 5;
2) Find a $\mathfrak{D}$-module $N_2 \subset M$ such that $N_1 \cap N_2 = 0$ and $N_1 + N_2 \underset{<\infty}{\subset} M$;
3) N_1^+ is a direct summand of M^+.

Proof : (1) $\Leftrightarrow$ (2): It is enough to apply the map $()^\perp$ to the conditions (12), take the quotient by $\mathfrak{B}^\perp$ and write $M = \mathfrak{D}^q / \mathfrak{B}^\perp$, $N_2 = \mathfrak{B}_b^\perp / \mathfrak{B}^\perp$ and $N_1 = \mathfrak{B}_a^\perp / \mathfrak{B}^\perp$.

(2) $\Rightarrow$ (3) Suppose $N_1 \oplus N_2 \underset{<\infty}{\subset} M$, then $N_1^+ \oplus N_2^+ \underset{<\infty}{\subset} M^+$, and by corollary 3.2 $N_1^+ \oplus N_2^+ = M^+$ because $N_1^+ \oplus N_2^+$ is free.

(2) $\Leftarrow$ (3) Suppose there exists N_2^+ with $N_1^+ \oplus N_2^+ = M^+$. Because $N_1 \subset M$ and $N_2 \cap M \subset M$ one has that $N_1 \oplus (N_2 \cap M) \underset{<\infty}{\subset} M$. □

From this theorem, the problem is, for given $N_1 \subset M$, to compute N_1^+ and M^+, and check whether N_1^+ is a direct summand of M^+. One way to do it is to construct a matrix R which maps a basis $e_1, e_2, \dots e_n$ of N_1^+ to the basis $f_1, f_2, \dots f_m$ of M^+. Thus, problem 5 is solvable is and only if the $n \times n$ minors of R generate the unit ideal $\mathfrak{D}$.

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